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$1/N_c$ corrections to the magnetic susceptibility of the QCD vacuum

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Abstract

We investigate the magnetic susceptibility of the QCD vacuum with the $1/N_c$ corrections taken into account, based on the instanton vacuum. Starting from the instanton liquid model we derive the gauged light-quark partition function in the presence of the current quark mass as well as of external Abelian vector and tensor fields. We consider the $1/N_c$ meson-loop corrections which are shown to contribute to the magnetic susceptibility by around 15 % for the up (and down) quarks. We also take into account the tensor terms of the quark-quark interaction from the instanton vacuum as well as the finite-width effects, both of which are of order $\mathcal{O}(1/N_c)$. The effects of the tensor terms and finite width turn out to be negligibly small. The final results for the up-quarks are given as: $\chi\langle i\psi^\dagger\psi\rangle_0 \simeq 35 - 40\,\mathrm{MeV}$ with the quark condensate $\langle i\psi^\dagger\psi\rangle_0$. We also discuss the pion mass dependence of the magnetic susceptibility in order to give a qualitative guideline for the chiral extrapolation of lattice data.

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I. INTRODUCTION

The QCD vacuum is known to be one of the most complicated states entangled with perturbative and strong nonperturbative fluctuations. In particular, the condensates of the quarks and gluons reveal their non-perturbative aspect and character. The quark condensate is taken as an order parameter associated with spontaneous chiral symmetry breaking (SχSB) which is probably one of the most important quantities for low-energy hadronic phenomena. Actually, the presence of an external source allows a more profound study of the QCD vacuum. For example, the constant electromagnetic field $F_{\mu\nu}$ induces another type of condensates leading to the nonzero magnetic susceptibility χ of the QCD vacuum defined as:

$$\langle \psi_f^\dagger \sigma_{\mu\nu} \psi_f \rangle_F = e_f \, \chi_f \, \langle i\psi_f^\dagger \psi_f \rangle_0 \, F_{\mu\nu}, \tag{1}$$

where e_f denotes the quark electric charge with the corresponding flavor f. It is natural to have the quark condensate $\langle i\psi_f^\dagger \psi_f \rangle_0$ in the chiral limit as a normalization factor in the right-hand side of Eq.(1), since SχSB is responsible for these quantities, i.e. the quark condensate and magnetic susceptibility. The value of χ_f was already predicted in the QCD sum rule and vector-dominance model [1, 2, 3, 4]: $\chi_u \langle i\psi_u^\dagger \psi_u \rangle_0 = 40 \sim 70\,\mathrm{MeV}$ at the scale of $1\,\mathrm{GeV}$, and in the instanton liquid model for the QCD vacuum [5] to $\sim 38\,\mathrm{MeV}$ and in Ref. [6] to $45 \sim 50\,\mathrm{MeV}$.[1] Moreover, Ref. [7] suggested that the magnetic susceptibility χ may be measured in the exclusive photoproduction of hard dijets $\gamma + N \to (\bar{q}q) + N$.

In the present work, we want to extend the previous work [6], incorporating the $1/N_c$ corrections [8, 9]. Since the instanton vacuum explains SχSB naturally via quark zero modes, it may provide a good framework to study the χ of the light-quark vacuum. Moreover, there are only two parameters in this approach, namely the average instanton size $\bar{\rho} \approx \frac{1}{3}\,\mathrm{fm}$ and average inter-instanton distance $\bar{R} \approx 1\,\mathrm{fm}$. The normalization scale of this approach can be defined by the average size of instantons and is approximately equal to $\rho^{-1} \approx 0.6\,\mathrm{GeV}$. The values of the $\bar{\rho}$ and $\bar{R}$ were estimated many years ago phenomenologically in Ref. [10] as well as theoretically in Ref. [11, 12, 13]. Furthermore, it was confirmed by various lattice simulations of the QCD vacuum [14, 15, 16]. Also lattice calculations of the quark propagator [17, 18] are in a remarkable agreement with that of Ref. [11]. A recent lattice simulation with the interacting instanton liquid model obtains $\bar{\rho} \simeq 0.32\,\mathrm{fm}$ and $\bar{R} \simeq 0.76\,\mathrm{fm}$ with the finite current quark mass m taken into account [19]. Considering the $1/N_c$ corrections, Ref. [9] found that the values of the instanton size and average distance should be readjusted as

$$\bar{\rho} = 0.35\,\mathrm{fm}, \quad \bar{R} = 0.86\,\mathrm{fm} \tag{2}$$

with $F_\pi = 88\,\mathrm{MeV}$ and $\langle i\psi^\dagger \psi \rangle_0 = (255\,\mathrm{MeV})^3$ required in the chiral limit. The values in Eq.(2) will be used in the following calculations.

In order to consider the $1/N_c$ corrections to the magnetic susceptibility of the QCD vacuum, we closely follow the formalism developed in Refs. [8, 9]. In order to calculate the magnetic susceptibility, we first have to find the light-quark partition function $Z[V, T, m]$ in the presence of the Abelian external vector V_μ and antisymmetric tensor $T_{\mu\nu}$ fields. So, we first calculate a zero-mode approximated quark propagator $\tilde{S}$ in the instanton ensemble and

[1] In the following we give all numbers for the up-quarks, which are due to assumed isospin invariance identical to the numbers for the down quarks. Thus we ignore the index u or f.

in the presence of these external fields. Using this propagator, we are able to compute the low-frequency part of the quark determinant for the non-zero m with the external Abelian vector V and tensor T fields. The relevant techniques have been already developed in Refs. [6].

The smallness of the packing parameter $\pi(\frac{\bar{\rho}}{R})^4 \approx 0.1$ makes it possible to average the determinant over collective coordinates of instantons with fermionic quasiparticles, i.e. constituent quarks ψ introduced. The averaged determinant turns out to be the light-quark partition function $Z[V,T,m]$ which is a functional of V and T and can be represented by a functional integral over the constituent quark fields with the gauged effective chiral action $S[\psi^\dagger,\psi,V,T]$. However, it is not trivial to make the action gauge-invariant due to the nonlocality of the quark-quark interactions generated by instantons. In the previous paper [20], it was demonstrated how to gauge the nonlocal effective chiral action in the presence of the external electromagnetic field and was shown that the low-energy theorem of the axial anomaly relevant to the process $G\tilde{G} \to \gamma\gamma$ is satisfied (see also Refs. [21, 22]). The gauged effective chiral action was also successfully applied to various observables of mesons and vacuum properties [6, 23, 24, 25].

The present work is organized as follows: In Section II, we review the general formalism for the magnetic susceptibility of the QCD vacuum. We first show how to derive from the instanton vacuum the gauged effective chiral action in the presence of the current quark mass as well as of external vector and tensor fields. We then derive the meson propagator for the meson-loop corrections to the magnetic susceptibility of the QCD vacuum, considering the fluctuation around the saddle-point. In Section III, we discuss the meson-loop corrections to the magnetic susceptibility, the contribution of the tensor quark-quark interactions to it, and the effects of the finite width of the instanton size, all of which are $1/N_c$ corrections to the magnetic susceptibility. In Section IV, we present the final results of the magnetic susceptibility of the QCD vacuum and discuss its relevance to the chiral extrapolation in lattice QCD, extending the results with larger pion masses employed. In the final Section, we summarize the present work and draw conclusions.

II. GENERAL FORMALISM

A. Light quark partition function in the presence of external vector and tensor fields

The light quark partition function $Z[V,T,m]$ with the quark mass m and external vector V_μ and tensor $T_{\mu\nu}$ fields is defined as

$$Z[V,T,m] = \int D\, A_\mu\, e^{-\frac{1}{4}G^2} \mathrm{Det}(i\partial\!\!\!/ + A\!\!\!/ + im + V\!\!\!/ + \sigma_{\mu\nu}T_{\mu\nu}), \tag{3}$$

where A_μ is the gluon field and $G_{\mu\nu}$ is the gluon field strength tensor. The basic assumption of the instanton liquid model is that one can evaluate the integral in a quasi-classical approximation, expanding it around the classical vacuum. The first evaluation of the partition function Eq.(3) was performed in Ref. [28, 29] in the absence of the external fields and in the chiral limit. The main purpose of the present paper is the extension of those results to the case of the nonzero quark mass m and external V, T fields, following the method given in [6, 9]. For this we split the quark determinant into the low- and high-frequency parts according $\mathrm{Det} = \mathrm{Det}_{\mathrm{low}}\mathrm{Det}_{\mathrm{high}}$ and concentrate on the evaluation of $\mathrm{Det}_{\mathrm{low}}$, which is

responsible for the low-energy domain. The high-energy part $\mathrm{Det}_{\mathrm{high}}$ is responsible mainly for the perturbative coupling renormalization.

We start with the zero-mode approximation for the propagator of a quark interacting with the i-th instanton. This propagator was considered in [26, 27, 28, 29]:

$$S_i = \frac{1}{\not{p} + \not{A}_i + im} = \frac{1}{\not{p}} + \frac{|\Phi_{i,0}\rangle\langle\Phi_{i,0}|}{im}. \tag{4}$$

While this zero approximation is good for small values of the current quark mass m, we need to extend it beyond the chiral limit as proposed in our previous works [6, 20, 30, 31] as follows:

$$S_i = S_0 + S_0\not{p}\frac{|\Phi_{0i}\rangle\langle\Phi_{0i}|}{c_i}\not{p}S_0, \tag{5}$$

where

$$c_i = -\langle\Phi_{0i}|\not{p}S_0\not{p}|\Phi_{0i}\rangle = im\langle\Phi_{0i}|S_0\not{p}|\Phi_{0i}\rangle = im\langle\Phi_{0i}|\not{p}S_0|\Phi_{0i}\rangle. \tag{6}$$

The approximation given in Eq.(5) allows us to project S_i to the correct zero-modes with the finite m:

$$S_i|\Phi_{0i}\rangle = \frac{1}{im}|\Phi_{0i}\rangle, \;\; \langle\Phi_{0i}|S_i = \langle\Phi_{0i}|\frac{1}{im}. \tag{7}$$

We can write the total quark propagator $\tilde{S}$ in the presence of the whole instanton ensemble A and external vector (V) and tensor (T) fields, and the quark propagator with a single instanton A_i as well as V and T as follows:

$$\tilde{S} = \frac{1}{\not{p} + \not{A} + \not{V} + \sigma_{\mu\nu}T_{\mu\nu} + im}, \;\; \tilde{S}_i = \frac{1}{\not{p} + \not{A}_i + \not{V} + \sigma_{\mu\nu}T_{\mu\nu} + im}. \tag{8}$$

Here, we assume that the total instanton field A may be approximated as a sum of the single instanton fields, $A = \sum_{i=1}^{N} A_i$, which is justified with the values of $\bar{\rho}$ and $\bar{R}$ in Eq.(2). With the instanton fields turned off, we write the quark propagator in the presence of the external fields and free quark propagator as follows:

$$\tilde{S}_0 = \frac{1}{\not{p} + \not{V} + \sigma_{\mu\nu}T_{\mu\nu} + im}, \quad S_0 = \frac{1}{\not{p} + im}. \tag{9}$$

We now expand the total quark propagator $\tilde{S}$ with respect to the single instanton field A_i:

$$\tilde{S} = \tilde{S}_0 + \sum_i(\tilde{S}_i - \tilde{S}_0) + \sum_{i\neq j}(\tilde{S}_i - \tilde{S}_0)\tilde{S}_0^{-1}(\tilde{S}_j - \tilde{S}_0) + \cdots. \tag{10}$$

The next step is to expand $\tilde{S}$ with respect to the external fields V and T, and express $\tilde{S}_i$ in terms of S_i. Since we use the zero-mode approximation, the expansion with respect to the vector field breaks the gauge invariance. In order to restore it, we introduce the following auxiliary field V' and gauge connection L_i:

$$\not{V}_i' = \bar{L}_i(\not{p} + \not{V})L_i - \not{p}. \tag{11}$$

The gauge connection L_i is defined as a path-ordered exponent

$$
\begin{aligned}
L_i(x,z_i) &= \mathrm{P}\exp\left(i\int_{z_i}^{x} dy_\mu V_\mu(y)\right), \\
\bar{L}_i(x,z_i) &= \gamma_0 L_i^\dagger(x,z_i)\gamma_0,
\end{aligned} \tag{12}
$$

where z_i denotes an instanton position. The field $V_i'(x,z_i)$ under flavor rotation $\psi(x) \to U(x)\psi(x)$ is transformed as $V_i'(x,z_i) \to U(z_i)V_i'(x,z_i)U^{-1}(z_i)$. The propagators $\tilde{S}_i$ and $\tilde{S}_0$ then have the following form:

$$
\begin{aligned}
\tilde{S}_i &= L_i S_i' \bar{L}_i, \quad S_i' = \frac{1}{\not{p} + \not{A}_i + \not{V}_i' + \sigma_{\mu\nu}T_{\mu\nu} + im}, \\
\tilde{S}_0 &= L_i S_{0i}' \bar{L}_i, \quad S_{0i}' = \frac{1}{\not{p} + \not{V}_i' + \sigma_{\mu\nu}T_{\mu\nu} + im},
\end{aligned} \tag{13}
$$

Expanding S_i' with respect to $\not{V}_i'$ and resumming it, we get

$$
S_i' = S_i(1 + \sum_n (-\hat{V}_i' S_i)^n) = S_{0i}' + S_{0i}'\not{p}\frac{|\Phi_{0i}\rangle\langle\Phi_{0i}|}{c_i - b_i}\not{p}S_{0i}', \tag{14}
$$

where

$$
\begin{aligned}
b_i &= \langle\Phi_{0i}|\not{p}(S_{0i}' - S_0)\not{p}|\Phi_{0i}\rangle, \\
c_i - b_i &= -\langle\Phi_{0i}|\not{p}S_{0i}'\not{p}|\Phi_{0i}\rangle.
\end{aligned} \tag{15}
$$

Rearranging Eq.(10) for the total propagator, we obtain

$$
\begin{aligned}
\tilde{S} &= \tilde{S}_0 + \tilde{S}_0 \sum_{i,j} \bar{L}_i^{-1}\not{p}|\Phi_{i0}\rangle \left(\frac{1}{-\mathcal{D}} + \frac{1}{-\mathcal{D}}\mathcal{C}\frac{1}{-\mathcal{D}} + \ldots\right)_{ij} \langle\Phi_{0j}|\not{p}L_j^{-1}\tilde{S}_0 \\
&= \tilde{S}_0 + \tilde{S}_0 \sum_{i,j} \bar{L}_i^{-1}\not{p}|\Phi_{i0}\rangle \left(\frac{1}{-\mathcal{V} - \mathcal{T}}\right)_{ij} \langle\Phi_{0j}|\not{p}L_j^{-1}\tilde{S}_0,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\mathcal{V}_{ij} &= \langle\Phi_{0i}|\not{p}(L_i^{-1}\tilde{S}_0\bar{L}_j^{-1})\not{p}|\Phi_{0j}\rangle - \langle\Phi_{0i}|\not{p}S_0 L_i^{-1}L_j\not{p}|\Phi_{0j}\rangle, \\
\mathcal{T}_{ij} &= (1-\delta_{ij})\langle\Phi_{0i}|\not{p}S_0 L_i^{-1}L_j\not{p}|\Phi_{0j}\rangle, \\
\mathcal{D}_{ij} &= \delta_{ij}\mathcal{V}_{ij} \equiv (b_i - c_i)\delta_{ij}, \\
\mathcal{C}_{ij} &= (1-\delta_{ij})\mathcal{V}_{ij}.
\end{aligned} \tag{17}
$$

We introduce now the modified zero-mode solution:

$$
|\phi_0\rangle = \frac{1}{\not{p}} L\not{p}|\Phi_0\rangle, \tag{18}
$$

which has the same chiral properties as the zero-mode solution $|\Phi_0\rangle$. Then we get

$$
\tilde{S} - \tilde{S}_0 = -\tilde{S}_0 \sum_{i,j} \not{p}|\phi_{0i}\rangle\langle\phi_{0i}| \left(\frac{1}{\mathcal{V} + \mathcal{T}}\right) |\phi_{0j}\rangle\langle\phi_{0j}|\not{p}\tilde{S}_0 \tag{19}
$$

with

$$\mathcal{V}+\mathcal{T}=\not{p}\tilde{S}_0\not{p}. \tag{20}$$

The final explicit form for Eq.(19) is written as

$$\mathrm{Tr}(\tilde{S}-\tilde{S}_0)=-\sum_{i,j}\langle\phi_{0,j}|\not{p}\,(\tilde{S}_0^2)\,\not{p}|\phi_{0,i}\rangle\langle\phi_{0,i}|\left(\frac{1}{\not{p}\tilde{S}_0\not{p}}\right)|\phi_{0,j}\rangle. \tag{21}$$

In order to derive the low-frequency part of the quark determinant, we now introduce a matrix operator $\tilde{B}(m)$ defined as follows:

$$\tilde{B}(m)_{ij}=\langle\phi_{0,i}|(\not{p}\tilde{S}_0\not{p})|\phi_{0,j}\rangle=\langle\Phi_{0i}|\not{p}\left(L_i^{-1}\,\tilde{S}_0\bar{L}_j^{-1}\right)\not{p}|\Phi_{0j}\rangle, \tag{22}$$

where i,j are indices for the different instantons. Then, we can show that

$$\begin{aligned}\ln\left(\mathrm{Det}_{\mathrm{low}}\right) &= \mathrm{Tr}\ln\left(\frac{i\not{\partial}+\not{A}+\not{V}+\sigma_{\mu\nu}T_{\mu\nu}+i\hat{m}}{i\not{\partial}+\not{V}+\sigma_{\mu\nu}T_{\mu\nu}+im}\right) \\ &= i\tilde{\mathrm{Tr}}\int^m dm'(\tilde{S}(m')-\tilde{S}_0(m')) \\ &= \sum_{i,j}\int^m dm'\frac{d\tilde{B}(m')_{ij}}{dm'}(\tilde{B}(m'))^{-1}_{ji}=\mathrm{Tr}\ln\tilde{B}(m),\end{aligned} \tag{23}$$

where $\tilde{\mathrm{Tr}}$ denotes the trace over the subspace of the quark zero modes. Thus, we have

$$\mathrm{Det}_{\mathrm{low}}[V,T,m]\cong\mathrm{Det}\tilde{B}(m), \tag{24}$$

where $\tilde{B}$ is the extension of Lee-Bardeen matrix B [27] in the presence of the external vector and tensor fields V and T. Taking m to be small and switching off the external fields, we can show that $\tilde{B}$ turns out to be the same as B to order $\mathcal{O}(m)$.

Averaging $\mathrm{Det}_{\mathrm{low}}$ over the instanton collective coordinates ξ, which provides the partition function $Z_N[V,T,m]$, we can obtain the fermionized representation of the partition function in Eq.(24):

$$\begin{aligned}Z_N[V,T,m] &= \langle\mathrm{Det}_{\mathrm{low}}[V,T,m]\rangle=\langle\mathrm{Det}\tilde{B}\rangle \\ &= \int D\psi D\psi^\dagger\exp\left(\int d^4x\psi^\dagger(\not{p}+\not{V}+\sigma_{\mu\nu}T_{\mu\nu}+im)\psi\right)\prod^{N_\pm}W_\pm[\psi^\dagger,\psi],\end{aligned} \tag{25}$$

where

$$\begin{aligned}W_\pm[\psi^\dagger,\psi] &= \int d^4\xi_\pm\prod_{N_f}\int d^4x\left(\psi^\dagger(x)\,\bar{L}^{-1}(x,\xi_\pm)\,\not{p}\Phi_{\pm,0}(x;\xi_\pm)\right) \\ &\times\int d^4y\left(\Phi^\dagger_{\pm,0}(y;\xi_\pm)(\not{p}\,L^{-1}(y,\xi_\pm)\psi(y)\right).\end{aligned} \tag{26}$$

The fermion fields $\psi^\dagger,\psi$ are interpreted as the dynamical quark fields or constituent quark fields induced by the zero-modes of the instantons.

Note that the partition function of Eq.(25) is invariant under local flavor rotations due to the gauge connection L in the interaction term $W_\pm[\psi^\dagger,\psi]$. While we preserve the gauge invariance of the effective chiral action by introducing the gauge connection, we have to pay a price: The effective action depends on the path in the gauge connection L. We will choose the straight-line path, though there is in general no physical reason why other choices should be excluded. However, in Refs. [6, 9] it was shown explicitly that for the magnetic susceptibility the path dependence does not come into play.

B. Chiral effective action in the presence of external vector and tensor fields

We are now in a position to derive the relevant partition function for the magnetic susceptibility of the QCD vacuum. As exposed in Refs. [6, 9] we first have to average the low-frequency part of the quark determinant $\mathrm{Det_{low}}$ over collective coordinates $\xi_\pm$. Here we assume a distribution of instanton sizes with vanishing width, i.e. of the form $d(\rho) = \delta(\rho - \bar{\rho})$. Having integrated over collective coordinates and having made a exponentiation, we reduce the partition function to the following form (for the case $N_f = 2$):

$$
\begin{aligned}
Z_N &= \int d\lambda_+ d\lambda_- D\bar{\psi} D\psi \exp(-\Gamma), && (27)\\
\Gamma &= N_\pm \ln \frac{K}{\lambda_\pm} - N_\pm + \psi^\dagger \left(i\not{\partial} + \not{V} + \sigma_{\mu\nu} T_{\mu\nu} + im \right) \psi + \lambda_\pm Y_2^\pm, && (28)\\
Y_2^\pm &= \alpha^2 \det_f J^\pm + \beta^2 \det_f J^\pm_{\mu\nu}, && (29)\\
\frac{\beta^2}{\alpha^2} &:= \frac{1}{8N_c} \frac{2N_c}{2N_c - 1} = \frac{1}{8N_c - 4} = \mathcal{O}\left(\frac{1}{N_c}\right), && (30)\\
J^\pm &= \psi^{\dagger\prime} \frac{1 \pm \gamma_5}{2} \psi', \\
J^\pm_{\mu\nu} &= \psi^{\dagger\prime} \sigma_{\mu\nu} \frac{1 \pm \gamma_5}{2} \psi' = \frac{1}{2} \left(J_{\mu\nu} \pm \frac{i}{2} \epsilon_{\mu\nu\rho\lambda} J_{\rho\lambda} \right), \\
\sigma_{\mu\nu} &= \frac{i}{2} \epsilon_{\mu\nu\rho\lambda} \sigma_{\rho\lambda}, && (31)\\
J_{\mu\nu} &= \psi^{\dagger\prime} \sigma_{\mu\nu} \psi', && (32)\\
\psi^{\dagger\prime} &= \psi^\dagger \bar{L}, \quad \psi' = L^{-1} \psi, && (33)
\end{aligned}
$$

where the determinant runs over the flavor space tacitly, and K is some inessential constant making the argument of the logarithm dimensionless. From Eq.(30) we immediately see that the contribution of the tensor terms is just one of the $1/N_c$-corrections. For simplicity, we will consider the tensor terms $J^\pm_{\mu\nu}$ later.

Having taken $N_+ = N_- = N/2$, having integrated over fermion fields [20, 28, 29, 31, 32], and having made a bosonization, we end up with the partition function in the presence of the external vector and tensor fields:

$$
Z_N[V, T, m] = \int d\lambda D\Phi \exp\left(-\Gamma[V, T, m, \lambda, \Phi]\right), \tag{34}
$$

where

$$
\Gamma[V, T, m, \lambda, \Phi] = N \ln \frac{K}{\lambda} - N + \Gamma_\Phi + \Gamma_\psi \tag{35}
$$

with

$$
\begin{aligned}
\Gamma_\Phi &= 2 \int d^4x \Phi^2, && (36)\\
\Gamma_\psi &= -\mathrm{Tr} \ln \left[\frac{\not{P} + \sigma_{\mu\nu} T_{\mu\nu} + im + i \frac{(2\pi\rho)^2 \sqrt{\lambda}}{2g} \bar{L} F \Phi \cdot \Gamma_\gamma F L}{\not{P} + \sigma_{\mu\nu} T_{\mu\nu} + im} \right]. && (37)
\end{aligned}
$$

Here, Φ denotes the meson fields $\Phi = \{\Phi_0, \boldsymbol{\Phi}\} = \{\sigma, \eta, \boldsymbol{\sigma}, \boldsymbol{\pi}\}$, and $\Phi^2 = \Phi_0^2 + \boldsymbol{\Phi}^2 = \sigma^2 + \eta^2 + \boldsymbol{\sigma}^2 + \boldsymbol{\pi}^2$. The $P_\mu = p_\mu + eV_\mu$, $p_\mu = i\partial_\mu$ and Tr stands for the functional trace, i.e. $\int d^4x \mathrm{tr}_c \mathrm{tr}_D\, tr_f$. The g is the color factor defined as $g^2 = 2N_c(N_c^2-1)/(2N_c-1)$. The $F(p)$ stands for the quark form factor generated by the fermionic zero modes. Though the explicit form of $F(p)$ is expressed in terms of the Bessel functions [28], we use in the present work the dipole type of the form factor for simplicity:

$$F(p) = \frac{\Lambda^2}{p^2+\Lambda^2}, \tag{38}$$

where Λ is the cut-off mass with $\Lambda \sim 1/\bar{\rho}$. In fact, the results with Eq.(38) are very similar to those with the original form factor [6].

In order to calculate the correlation functions, we follow the general effective action approaches [33, 34]. The effective partition function is now written as

$$Z_N[V,T,m] = \exp(-\Gamma_{\rm eff}[V,T,m,\lambda,\langle\Phi\rangle_0]) \tag{39}$$

with the effective action $\Gamma_{\rm eff}[V,T,m,\lambda,\langle\Phi\rangle_0]$, where vacuum average of the fields $\langle\Phi\rangle_0$ must be taken as a solution of the vacuum:

$$\frac{\partial}{\partial\langle\Phi\rangle}\Gamma_{\rm eff}[V,T,m,\lambda,\langle\Phi\rangle_0] = 0 \tag{40}$$

and of the coupling λ from the saddle-point condition:

$$\frac{d}{d\lambda}\Gamma_{\rm eff}[V,T,m,\lambda,\langle\Phi\rangle_0] = \frac{\partial}{\partial\lambda}\Gamma_{\rm eff}[V,T,m,\lambda,\langle\Phi\rangle_0] = 0. \tag{41}$$

C. Vacuum in the presence of external vector and tensor fields

The instanton vacuum naturally realizes SχSB, which is measured by the quark condensate $\langle i\psi_f^\dagger\psi_f\rangle_0$. The strong quark-quark interaction generated by the instanton vacuum (26) brings about the non-zero vacuum expectation value of the scalar-isoscalar meson field, i.e. $\sigma = \langle\Phi_0\rangle_0$, which can be obtained by minimizing the effective action:

$$\frac{\partial\Gamma_{\rm eff}[m,\lambda,\sigma]}{\partial\sigma} = 0, \quad \frac{\partial\Gamma_{\rm eff}[m,\lambda,\sigma]}{\partial\lambda} = 0. \tag{42}$$

Expanding the effective action in one-loop meson approximation, we obtain the following expression:

$$\Gamma_{\rm eff}[m,\lambda,\sigma] = \Gamma_0[m,\lambda,\sigma] + \Gamma_{\rm eff}^{\rm mes}[m,\lambda,\sigma], \tag{43}$$

where $\Gamma_{\rm eff}^{\rm mes}[m,\lambda,\sigma]$ represents one-loop meson contribution and is expressed as follows:

$$\Gamma_{\rm eff}^{\rm mes}[m,\lambda,\sigma] = \frac{1}{2}\mathrm{Tr}\ln\left(\left.\frac{\delta^2\Gamma_0[m,\lambda,\Phi]}{\delta\Phi_i\delta\Phi_j}\right|_\sigma\right). \tag{44}$$

Here the Γ_0 is just identical to the effective action in Eq.(35). In Ref. [9], it is extensively studied how to solve Eqs.(42) and (44).

We now find the vacuum average of the meson fields and coupling λ in the presence of external fields V and T. Though, in fact, their influence appears from the second order, it is enough to consider them to the first order in order to calculate the correlation functions. Thus, we can safely use σ and λ derived from Ref. [9]. Then, we can straightforwardly derive the meson-loop effective action in the presence of the vector and tensor fields similar to Eq. (44):

$$\Gamma_{\rm eff}[V,T,m,\lambda,\sigma] = \Gamma_0[V,T,m,\lambda,\sigma] + \Gamma_{\rm eff}^{\rm mes}[V,T,m,\lambda,\sigma], \tag{45}$$

where

$$\Gamma_{\rm eff}^{\rm mes}[V,T,m,\lambda,\sigma] = \frac{1}{2}{\rm Sp}\ln\left[4\delta_{ij} - \frac{1}{\sigma^2}{\rm Tr}\frac{1}{\not{P}+\sigma_{\mu\nu}T_{\mu\nu}+i\mu(P)} \times \Gamma_i M(P)\frac{1}{\not{P}+\sigma_{\mu\nu}T_{\mu\nu}+i\mu(P)}\Gamma_j M(P)\right] \tag{46}$$

with $\mu(P) = m + M(P)$ and $M(P) = MF^2(P)$. The M denotes the dynamical quark mass defined as

$$M = \frac{(2\pi\rho)^2\sqrt{\lambda}}{2g}\sigma. \tag{47}$$

The Sp stands for the trace over the meson degrees of freedom.

We have to keep for the self-consistency of the one-meson-loop approximation the meson propagators in the leading order (without meson loops) from the equation:

$$\frac{\partial\Gamma_0[m,\lambda,\sigma]}{\partial\sigma} = 0. \tag{48}$$

Then, the meson propagator can be redefined as

$$\Pi_i(q) \Rightarrow \tilde{\Pi}_i(q) = \frac{\sigma^2\mathbf{V}_0}{{\rm Tr}\mathcal{Q}(p) + {\rm Tr}\left(\mathcal{Q}(p)\Gamma_i\mathcal{Q}(p+q)\Gamma_i\right)}. \tag{49}$$

III. MESON-LOOP CORRECTIONS TO THE MAGNETIC SUSCEPTIBILITY OF THE QCD VACUUM

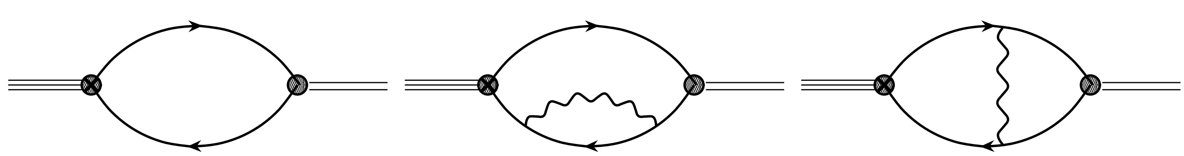

FIG. 1: The diagrams which contribute to the magnetic susceptibility in the leading order (first) and next-to-leading order (NLO) in $1/N_c$. The triple line on the left corresponds to the local tensor current $\bar\psi\sigma_{\mu\nu}\psi$, the double line on the right represents the vector current $\bar\psi\gamma_\mu\psi$. The interaction of the vector current with quarks contains both the local and nonlocal terms (see Appendix A and B).

We are now ready for the calculation of the magnetic susceptibility χ from the instanton vacuum, taking into account the meson-loop corrections. Diagrammatically, we consider three different contributions as shown in Fig. 1. The first diagram in Fig. 1, which is the leading-order (LO) contribution in the large N_c expansion, has been already calculated in Ref. [6]:

$$\chi\langle i\psi^\dagger\psi\rangle_0 = 4N_c\int\frac{d^4p}{(2\pi)^4}\left[\frac{\mu(p)}{(p^2+\mu(p)^2)^2}-\frac{m}{(p^2+m^2)^2}\right]-4N_c\int\frac{d^4p}{(2\pi)^4}\frac{pM(p)F'(p)}{(p^2+\mu(p)^2)^2}, \quad (50)$$

where the quark condensate in the chiral limit plays the role of the normalization. The infrared region of the first integral in Eq.(50) brings out the contribution of order $\mathcal{O}(m\ \ln\ m)$ which is almost model-independent. In order to calculate the magnetic susceptibility explicitly, we first consider two different regions while integrating: $0 < p < 1\,\mathrm{GeV}$ and $1\,\mathrm{GeV} < p < \infty$. Then we are able to evaluate the part of the free quark analytically. The second integral in Eq. (50) is related to the nonlocal contribution without which the vector current is not conserved, i.e. the Ward-Takahashi identity is broken. It arises from the nonlocal quark-quark interaction and is called *nonlocal current.*

The meson-loop corrections can be derived by differentiating the mesonic effective action given in Eq.(46) with respect to the external tensor field:

$$\left.\frac{\delta\Gamma^{\mathrm{mes}}_{\mathrm{eff}}}{\delta T_{\mu\nu}(z)}\right|_{T=0} = -\left(\frac{M_0}{\sigma}\right)^2\sum_i\int d^4xd^4y\,\Pi_i(x-y)\mathrm{Tr}\left[\left(F(P)\frac{1}{\not{P}+i\mu(P)}\right)_{yz}\sigma_{\mu\nu}\right.$$
$$\left.\times\left(\frac{1}{\not{P}+i\mu(P)}F(P)\right)_{zx}\Gamma_i\left(F(P)\frac{1}{\not{P}+i\mu(P)}F(P)\right)_{xy}\Gamma_i\right], \quad (51)$$

which corresponds to the second and third diagrams in Fig. 1. Note that here the order of the operators should be kept properly, since $[P_\mu, P_\nu] = ieF_{\mu\nu}$. Using the Schwinger method [35, 36], we continue to calculate Eq.(51). Having carried out a laborious but straightforward calculation, we end up with the expression for the meson-loop corrections to the magnetic susceptibilities:

$$\chi\langle i\psi^\dagger\psi\rangle_0^{\mathrm{mes}} = \frac{1}{2}\sum_i\int\frac{d^4q}{(2\pi)^4}V_i^\chi(q)\tilde{\Pi}_i(q), \quad (52)$$

where V_i^χ are the vertex functions which are given explicitly in Appendix A and B. The $\Pi_i(q)$ denote the meson propagators given in Eq.(49). The final expression for the magnetic susceptibility is now written as

$$\chi\langle\bar\psi\psi\rangle = 4N_c\int\frac{d^4p}{(2\pi)^4}\frac{\mu(p)-pMF(p)F'(p)}{(p^2+\mu^2(p))^2} -$$
$$4N_c\int\frac{d^4p}{(2\pi)^4}\frac{m}{(p^2+m^2)^2}+\frac{1}{2}\sum_i\int\frac{d^4q}{(2\pi)^4}V_i^\chi(q)\tilde{\Pi}_i(q). \quad (53)$$

The numerical result for Eq.(53) is finally obtained as follows:

$$\begin{aligned}\chi\langle\bar\psi\psi\rangle_0 &= N_c\left[0.015\,\mathrm{GeV}+5.3\times10^{-4}\left(\frac{m}{\mathrm{GeV}}\right)+\frac{m}{2\pi^2}\ln\left(\frac{m}{\mathrm{GeV}}\right)\right]\\ &- \left[0.007\,\mathrm{GeV}-\left(\frac{0.415m}{\mathrm{GeV}}\right)-\left(\frac{0.198m}{\mathrm{GeV}}\right)\ln\left(\frac{m}{\mathrm{GeV}}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+\ \mathcal{O}\left(m^2, \frac{1}{N_c^2}\right) \\
&=\ 0.038\,\mathrm{GeV} - \left(\frac{0.413m}{\mathrm{GeV}}\right) - \left(\frac{0.0462m}{\mathrm{GeV}}\right) \ln\left(\frac{m}{\mathrm{GeV}}\right) \\
&+\ \mathcal{O}\left(m^2, \frac{1}{N_c^2}\right).
\end{aligned} \tag{54}
$$

IV. CONTRIBUTION OF THE TENSOR TERMS TO THE MAGNETIC SUSCEPTIBILITY OF THE QCD VACUUM

Since the meson-loop corrections, however, are of order $\mathcal{O}(1/N_c)$ in the large N_c expansion, we have to take into account the tensor terms in Eq.(29) which are also of $1/N_c$ order. It is rather tedious to consider the tensor terms [9]. In the present Section, we want to show briefly how to associate with the tensor terms for the magnetic susceptibility. The details can be found in Ref. [9].

Introducing the tensor meson fields, the bosonized effective partition function can be written as follows:

$$
Z_N[V,T,m] = \int d\lambda d\lambda D\Phi D\Phi_{\mu\nu} \exp(-\Gamma[V,T,m,\lambda,\Phi,\Phi_{\mu\nu}]), \tag{55}
$$

where

$$
\begin{aligned}
\Gamma\ &=\ -N_{\pm}\ln\lambda_{\pm} + 2\left(\Phi_i^2 + \frac{1}{2}\Phi_{\mu\nu,i}^2\right) \\
&-\ \mathrm{Tr}\ln\left[\not{P} + T_{\mu\nu}\sigma_{\mu\nu} + im + i\sqrt{\lambda}\bar{L}F(p)\left(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{\mu\nu,i}\sigma_{\mu\nu}\Gamma_i\right)F(p)L^{-1}\right].
\end{aligned} \tag{56}
$$

Then, the gap equation and meson fields are changed as follows:

$$
\frac{N}{\mathbf{V}_0} = \frac{1}{2}\mathrm{Tr}\left(\frac{i\sqrt{\lambda}\bar{L}F(p)(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{\mu\nu,i}\sigma_{\mu\nu}\Gamma_i)F(p)L^{-1}}{\not{P} + im + i\sqrt{\lambda}\bar{L}F(p)(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{\mu\nu,i}\sigma_{\mu\nu}\Gamma_i)F(p)L^{-1}}\right), \tag{57}
$$

$$
\Phi_i = \frac{1}{4}\mathrm{Tr}\left(\frac{i\sqrt{\lambda}\bar{L}F(p)\alpha\Gamma_i F(p)L^{-1}}{\not{P} + im + i\sqrt{\lambda}\bar{L}F(p)(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{\mu\nu,i}\sigma_{\mu\nu}\Gamma_i)F(p)L^{-1}}\right), \tag{58}
$$

$$
\Phi_{\mu\nu,i} = \frac{1}{4}\mathrm{Tr}\left(\frac{i\sqrt{\lambda}\bar{L}F(p)\beta\sigma_{\mu\nu}\Gamma_i F(p)L^{-1}}{\not{P} + im + i\sqrt{\lambda}\bar{L}F(p)(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{\lambda\rho,i}\sigma_{\lambda\rho}\Gamma_i)F(p).L^{-1}}\right) \tag{59}
$$

In general, it is of great difficulty to evaluate Eqs.(57)-(59). However, if V_μ varies very slowly, the calculation becomes much simpler. Indeed, for $V_\mu = \text{const.}$ in Eq.(59), we get, respectively,

$$
\Phi_{\mu\nu} \approx c_1 i F_{\mu\nu} + \mathcal{O}(V^3). \tag{60}
$$

The constant c_1 is proportional to the coupling of the tensor meson to the vector current V_μ, which is just the inverse of the tensor-meson propagator, as will be shown below. Then,

we can show that saddle-point values of λ and σ acquire only $\mathcal{O}(V^2)$-order corrections, but not $\mathcal{O}(V)$. Substituting these results into the effective action Eq.(56), one can easily get

$$\frac{\partial\Gamma}{\partial T_{\mu\nu}}\bigg|_{T=0} = \mathrm{Tr}\left(\sigma_{\mu\nu}\frac{1}{\not{P}+im+iM\bar{L}F(p)(1+\frac{\beta}{2\alpha\sigma}\Phi_{\lambda\rho,i}\sigma_{\lambda\rho}\Gamma_i)F(p)L^{-1}}\right) \tag{61}$$

with

$$\sigma = \sqrt{\frac{N}{2\mathbf{V}_0}} + \mathcal{O}(V^2). \tag{62}$$

It is then straightforward but tedious to calculate the contribution of the tensor terms to the magnetic susceptibility:

$$\chi\langle i\psi^\dagger\psi\rangle_0 = 4N_c\int\frac{d^4p}{(2\pi)^4}\frac{\mu(p)-pMF(p)F'(p)}{(p^2+\mu^2(p))^2} + 4N_c\frac{\beta c_1}{\alpha\sigma}\int\frac{d^4p}{(2\pi)^4}\frac{MF^2(p)\mu^2(p)}{(p^2+\mu^2(p))^2} \tag{63}$$

with

$$c_1 = \frac{2N_c\frac{\beta}{\alpha\sigma}}{\left(1-2N_c\left(\frac{\beta}{\alpha\sigma}\right)^2\int\frac{d^4p}{(2\pi)^4}\frac{M^2F^4(p)\mu^2(p)}{(p^2+\mu^2(p))^2}\right)}\int\frac{d^4p}{(2\pi)^4}\frac{MF^2(p)(\mu(p)-pMF(p)F'(p))}{(p^2+\mu^2(p))^2}. \tag{64}$$

In Fig. 2, the contribution of the tensor terms is drawn in the dashed box.

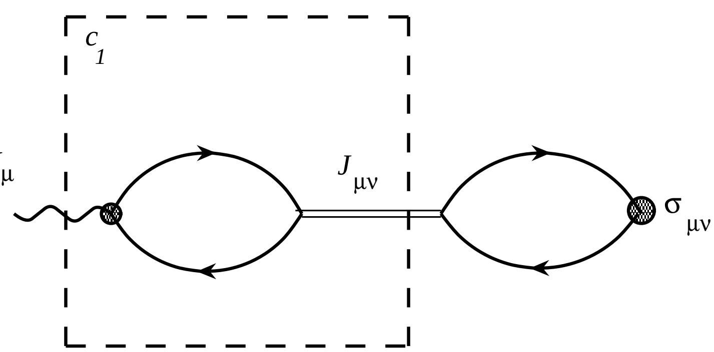

FIG. 2: The Feynman diagram for the tensor contribution.

The numerical results for the contribution of the tensor term are as follows:

$$\chi\langle i\psi^\dagger\psi\rangle_0^{\mathrm{Tensor}} = 0.28\times10^{-3}\,\mathrm{GeV}, \quad \frac{\chi\langle i\psi^\dagger\psi\rangle_0^{\mathrm{Tensor}}}{\chi\langle i\psi^\dagger\psi\rangle_{LO}} \simeq 0.006. \tag{65}$$

Thus, the contribution of the tensor terms to the magnetic susceptibility turns out to be below 1 %, so that we safely neglect them.

V. FINITE-WIDTH EFFECTS ON THE MAGNETIC SUSCEPTIBILITY OF THE QCD VACUUM

We have assumed so far that the width of the instanton-size distribution can be approximated to zero, i.e.:

$$d(\rho) = \delta(\rho - \bar{\rho}), \tag{66}$$

where all instantons have the same size $\bar{\rho}$. This approximation is justified in the large N_c limit as shown in Refs. [11, 28]:

$$\frac{\langle \rho^2 \rangle - \langle \rho \rangle^2}{\langle \rho \rangle^2} \sim \mathcal{O}\left(\frac{1}{N_c}\right). \tag{67}$$

Since we, however, consider the $1/N_c$ corrections to the magnetic susceptibility of the QCD vacuum, we also have to take into account the effects of the finite width, which are of order $\mathcal{O}(1/N_c)$ as well. For the numerical evaluation, we take the value for the fluctuation of the width as

$$\delta\rho^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2 \approx \frac{0.56}{N_c}\,\mathrm{GeV}^{-2}, \tag{68}$$

which arises from the two-loop size distribution of Ref.[11]. In order to take into account the effects of the finite width, we first return to Eq. (26), in the evaluation of which Eq. (66) was assumed. The integration over ρ does not change the exponentiation procedure so that we still can obtain the standard $2N_f$ quark-quark interaction in the effective action Γ. However, we should slightly modify the standard procedure to carry out the bosonization. For any function $J(z,\rho)$ we have the identity

$$\exp\left(-\int d^4z d\rho J^2(z,\rho)\right) = \int D\Phi \exp\left(-\frac{1}{4}\int d^4z d\rho\, \Phi^2 + \int d^4z d\rho\, \Phi J\right). \tag{69}$$

Thus, the effective action is written as

$$\Gamma \;=\; \frac{N}{\mathbf{V}_0}\ln\lambda + 2\int d^4z d\rho\, \Phi^2(z,\rho) \tag{70}$$

$$-\;\mathrm{Tr}\ln\left(\not{P} + \sigma_{\mu\nu}T_{\mu\nu} + im + ic\sqrt{\lambda}\int d^4z d\rho \hat{K}(z,\rho)\Phi(z,\rho)\right), \tag{71}$$

where c is an inessential constant and $\hat{K}_{x,y}(z,\rho) \simeq \bar{\phi}(x-z,\rho)\phi(z-y,\rho)$. The LO gap equations are then expressed as

$$\frac{N}{\mathbf{V}_0} \;=\; \frac{1}{2}\mathrm{Tr}\left(\frac{ic\sqrt{\lambda}\int d^4z d\rho \Phi(\rho)\hat{K}(z,\rho)}{(\not{p} + im + ic\sqrt{\lambda}\int d^4z d\rho \hat{K}(z,\rho)\Phi(\rho)}\right), \tag{72}$$

$$\Phi_0(\rho) \;=\; \frac{1}{4}\mathrm{Tr}\left(\frac{ic\sqrt{\lambda}\hat{K}(z,\rho)}{\not{p} + im + ic\sqrt{\lambda}\int d^4z d\rho \hat{K}(z,\rho)\Phi_0(\rho)}\right), \tag{73}$$

where $\Phi_0(\rho)$ has the quantum numbers of the σ. The unknown variables in Eqs.(72,73) are the parameter λ and the function $\Phi(\rho)$. In general, these are very complicated equations

but can be solved numerically. However, when the width is small, we can expand them with respect to $\delta\rho^2$. In particular, we expand the functions $\Phi(\rho)$ and $\hat{K}(\rho)$ with respect to $(\rho-\bar{\rho})$, so that we obtain a system of equations for the Taylor coefficients Φ_i and λ. Solving it, we derive the function $\Phi(\rho)$ (at least in the vicinity of $\bar{\rho}$). Details of the evaluation can be found in Ref. [9]. Finally, we are able to evaluate the corrections of the finite width of the instanton size:

$$(\chi\langle\bar{\psi}\psi\rangle_0)^{\rm FW} = \left(-0.00033 + 0.011\left(\frac{m}{\rm GeV}\right) - 0.096\left(\frac{m}{\rm GeV}\right)^2\right). \tag{74}$$

The numerical calculation of Eq.(74) turns out to be very tiny. The effects of the finite width is about 0.6%. Thus, we can safely neglect these effects on the magnetic susceptibility.

VI. RESULTS AND DISCUSSION

We have calculated the magnetic susceptibility $\chi\langle i\psi^\dagger\psi\rangle_0$ as a function of the current quark mass, with the form factor in Eq. (38) employed. For the further discussion it is more appropriate to plot the magnetic susceptibility in terms of the square of the pion mass, m_π^2. Following Ref. [9] both quantities are directly related as

$$\begin{aligned} m_\pi^2 &= m\left(\left(3.49+\frac{1.63}{N_c}\right)+m\left(15.5+\frac{18.25}{N_c}+\frac{13.5577}{N_c}\ln m\right)+\mathcal{O}(m^2)\right) \\ &= m\left(4.04\ +21.587\,m+4.52\,m\ln m+\mathcal{O}(m^2)\right)[\mathrm{GeV}^2]. \end{aligned} \tag{75}$$

Figure 3 shows the magnetic susceptibility $\chi\langle i\psi^\dagger\psi\rangle_0$ as a function of the square of the pion mass, m_π^2, with the form factor in Eq. (38) employed. The long-dashed curve is the result of the leading-order contribution in the large N_c expansion, which is the same as that in Ref. [6]. The short-dashed curve depicts the meson-loop corrections expressed in Eq.(52). It is negative for all values of m_π^2. Moreover, since the chiral-log term in the leading-order contribution is almost cancelled by the meson-loop corrections, the total magnetic susceptibility depends on m_π^2 almost linearly, as shown in Fig. 3.

The meson-loop corrections suppress the magnetic susceptibility by about 15 % near the physical value of $m_\pi^2 = (0.14\,\mathrm{GeV})^2 = 0.02\ \mathrm{GeV}^2$. This fact implies that the large N_c expansion seems to be reliable for them. As a result, the magnetic susceptibility for the up and down quarks is: $\chi_{u,d} = 35 \sim 40$ MeV. As for the up and down quarks, the present result is comparable to that of Ref. [2, 3, 4].

Actually, the effect of the meson-loop corrections depends on m_π^2. As one can see in Fig. 3 at $m_\pi^2 \sim 0.4\ \mathrm{GeV}^2$ they bring the leading order term down to almost 50%. Thus it makes no sense to calculate the susceptibility for larger values of m_π^2. Even this limited range given in Fig. 3 might be interesting for practitioners of lattice gauge QCD calculations. Those calculations suffer generally under the fact that by technical reasons they cannot be performed with current quark masses corresponding to the physical pion mass. Usually they have to use $m_\pi^2 > (0.4 \sim 0.6)\,\mathrm{GeV}^2$. In order to obtain a physical observable one has to extrapolate from the m_π^2 used in the lattice calculation to $m_\pi^2 = (0.14\,\mathrm{GeV})^2$. There exist different extrapolation techniques as e.g. linear extrapolation, extrapolation by chiral perturbation theory [37, 38, 39, 40, 41], finite range extrapolation [42, 43], extrapolation by suitable chiral models [44, 45, 46], etc. Apparently the present calculations suggest in Fig. 3 that for the magnetic susceptibility of the vacuum a simple linear extrapolation might be sufficient.

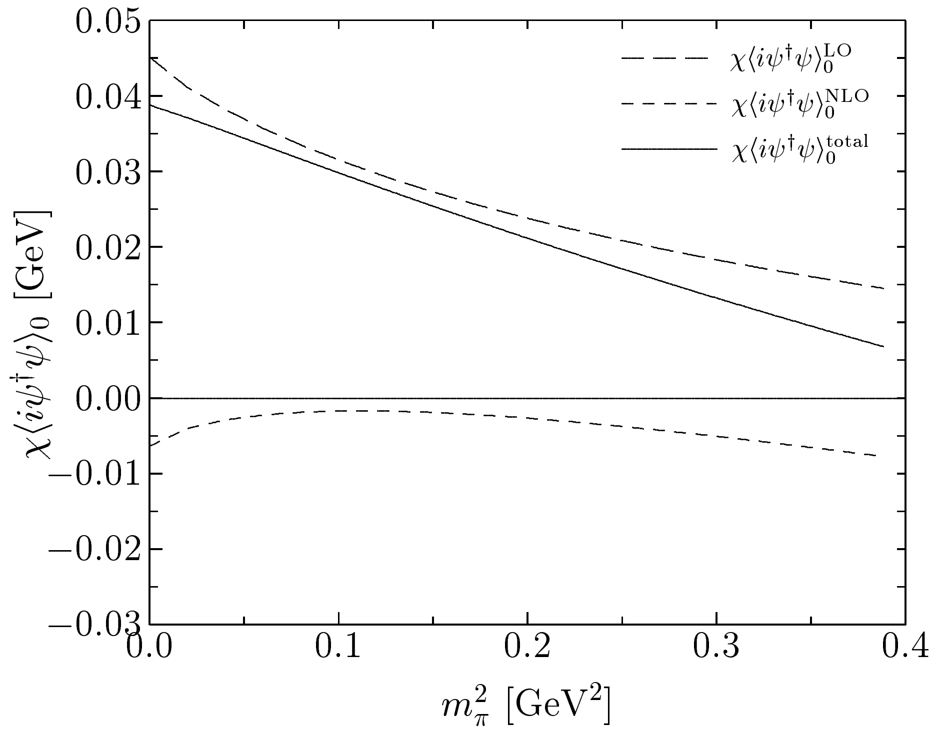

FIG. 3: The dependence of the magnetic susceptibility $\chi\langle i\psi^\dagger\psi\rangle_0$ on the square of the pion mass, m_π^2. The long-dashed curve is a contribution of the leading order result (LO), the short-dashed curve is that of the $1/N_c$ -correction (NLO), and the solid curve is the total contribution. Note that in the given range the total result depends on m_π^2 almost linearly: The chiral-log terms in the leading-order and next-to-leading order results almost cancel each other.

VII. SUMMARY AND CONCLUSION

In the present work, we have investigated the magnetic susceptibility of the QCD vacuum, based on the low-energy effective QCD partition function from the instanton vacuum. We first have constructed the effective partition function in the presence of the external vector and tensor fields as well as of the current quark mass. We have made the effective action gauge-invariant at the outset. We also have considered the meson-loop corrections which are a part of the $1/N_c$ corrections. To be consistent, we have taken into account the effects of the finite width of the instanton size distribution as well as the tensor terms of the quark-quark interactions, since they are also of order $\mathcal{O}(1/N_c)$. However, both these effects turn out to be negligibly small.

The meson-loop corrections contribute to the magnetic susceptibility χ negatively and their chiral-log terms almost cancel those of the leading-order contribution. They contribute to the magnetic susceptibility for the up and down quarks by around 15 %. As a result, assuming for the up quarks (and identically for the down quarks) $m_{u,d} \simeq 5\,\mathrm{MeV}$ we obtain the magnetic susceptibility $\chi\langle i\psi^\dagger\psi\rangle_0 = 35 \sim 40\,\mathrm{MeV}$. These results are comparable to the estimates of the vector dominance and of the QCD sum rule approaches [2, 3, 4]. It turns out that χ is almost a linearly decreasing function of the current quark mass m or of m_π^2. To know this might be useful for the chiral extrapolation of lattice data for the magnetic susceptibility of the QCD vacuum.

Acknowledgments

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APPENDIX A: SCALAR VERTICES

We present the explicit expression for the vertex function for the isoscalar scalar field $V_\sigma^\chi(k)$. The vertex function for the isovector scalar field $V_{\boldsymbol{\sigma}}^\chi(k)$ is given as $V_{\boldsymbol{\sigma}}^\chi(k) = -3V_\sigma^\chi(k)$. The $V_\sigma^\chi(k)$ is written as the following expression:

$$\begin{aligned}
V_\sigma^\chi(k) \;=\; & N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2F(p)F(k+p)}{3p\,(p^2+\mu(p)^2)^3 \left((k+p)^2+\mu\,(k+p)^2\right)^2} \Big[12pF(p)F(k+p)\left((k+p)^2+\mu\,(k+p)^2\right) \\
& \times \big(-3\mu(p)p^2-\mu(k+p)p^2+2MF(p)\left(p^2-\mu(p)\mu(k+p)+k\cdot p\right)F'(p)p \\
& +2\mu(p)^2\mu(k+p)-3\mu(p)k\cdot p\big)+\frac{2F(p)F(k+p)\left(p^2+\mu(p)^2\right)}{(k+p)}\big(-3\mu(k+p)p^4-3k\mu(k+p)p^3 \\
& -6\mu(k+p)k\cdot pp^2-6k\mu(k+p)k\cdot pp+M\big((k+p)F(p)\big(-3p^4+2k^2p^2-6\mu(k+p)^2p^2 \\
& -9k\cdot pp^2-8(k\cdot p)^2\big)F'(p)-pF(k+p)\left(3p^4+k^2p^2+6k\cdot pp^2+2(k\cdot p)^2\right)F'(k+p)\big) \\
& +\mu(p)\big(-3p^4-3kp^3-3k\cdot pp^2-6M(k+p)F(p)\mu(k+p)F'(p)p^2+6(k+p)\mu(k+p)^2p \\
& -3kpk\cdot p+4M^2F(p)F(k+p)\left(k^2p^2-(k\cdot p)^2\right)F'(p)F'(k+p)\big) \\
& -3(k+p)\mu(p)^2\Big(M\Big(F(p)\left(p^2+k\cdot p\right)F'(p)+p\sqrt{(k+p)^2}F(k+p)F'(k+p)\Big)-2p\mu(k+p)\Big)\Big) \\
& +4\left(p^2+\mu(p)^2\right)\left((k+p)^2+\mu\,(k+p)^2\right)\big(-p\,(3\mu(p)-2MpF(p)F'(p)) \\
& \times\Big(\frac{1}{p^3}\big(F(k+p)\big(pF''(p)(k\cdot p)^2+\left(k^2p^2+k\cdot pp^2-(k\cdot p)^2\right)F'(p)\big)\big) \\
& -\frac{1}{(k+p)^3}\Big(F(p)\Big((k+p)\left(k^2+k\cdot p\right)^2F''(k+p)-\left(k^4+3(k\cdot p)^2+\left(3k^2+p^2\right)k\cdot p\right)F'(k+p)\Big)\Big)\Big) \\
& -\left(3p\mu(p)+3p\mu(k+p)+2MF(p)k\cdot pF'(p)\right)\big(F(k+p)\left(pF'(p)+k\cdot pF''(p)\right) \\
& -\frac{1}{(k+p)^3}\big(F(p)\big((k+p)(k^2+k\cdot p)(p^2+k\cdot p)F''(k+p) \\
& -\left(p^4+2k^2p^2+(k\cdot p)^2+\left(k^2+3p^2\right)k\cdot p\right)F'(k+p)\big)\big)\big)\big)\Big]\,.
\end{aligned} \tag{A1}$$

APPENDIX B: PSEUDOSCALAR VERTICES

We present the explicit expression for the vertex function of the pseudoscalar isoscalar field $V_\eta^\chi(k)$. The vertex function of the pseudoscalar isovector field $V_{\boldsymbol{\pi}}^\chi(k)$ is given as $V_{\boldsymbol{\pi}}^\chi(k) = -3V_\eta^\chi(k)$. The $V_\eta^\chi(k)$ is written as the following expression:

$$V_\eta^\chi(k) \;=\; N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^2F(p)F(k+p)}{3p\,(p^2+\mu(p)^2)^3 \left((k+p)^2+\mu\,(k+p)^2\right)^2} \Big[-12pF(p)F(k+p)\left((k+p)^2+\mu\,(k+p)^2\right)$$

$$
\begin{aligned}
&\left(-3\mu(p)p^2+\mu(k+p)p^2+2MF(p)\left(p^2+\mu(p)\mu(k+p)+k\cdot p\right)F'(p)p-2\mu(p)^2\mu(k+p)-3\mu(p)k\cdot p\right)\\
&+\frac{2F(p)F(k+p)\left(p^2+\mu(p)^2\right)}{(k+p)}\left(3\mu(k+p)p^4+3k\mu(k+p)p^3+6\mu(k+p)k\cdot pp^2+6k\mu(k+p)k\cdot pp\right.\\
&+M\left((k+p)F(p)\left(-3p^4+2k^2p^2-6\mu(k+p)^2p^2-9k\cdot pp^2-8(k\cdot p)^2\right)F'(p)\right.\\
&\left.+\,pF(k+p)\left(3p^4+k^2p^2+6k\cdot pp^2+2(k\cdot p)^2\right)F'(k+p)\right)-\mu(p)\left(3p^4+3kp^3+3k\cdot pp^2\right.\\
&-6M(k+p)F(p)\mu(k+p)F'(p)p^2-6(k+p)\mu(k+p)^2p+3kk\cdot pp\\
&\left.\left.+4M^2F(p)F(k+p)\left(k^2p^2-(k\cdot p)^2\right)F'(p)F'(k+p)\right)\right.\\
&+3(k+p)\mu(p)^2\left(M\left(p(k+p)F(k+p)F'(k+p)-F(p)\left(p^2+k\cdot p\right)F'(p)\right)-2p\mu(k+p)\right))\\
&+4\left(p^2+\mu(p)^2\right)\left((k+p)^2+\mu\left(k+p\right)^2\right)\\
&\left(p\left(3\mu(p)-2MpF(p)F'(p)\right)\left(\frac{1}{p^3}\left(F(k+p)\left(pF''(p)(k\cdot p)^2+\left(k^2p^2+k\cdot pp^2-(k\cdot p)^2\right)F'(p)\right)\right)\right.\right.\\
&\left.\left.-\frac{1}{(k+p)^3}\left(F(p)\left((k+p)\left(k^2+k\cdot p\right)^2F''(k+p)-\left(k^4+3(k\cdot p)^2+\left(3k^2+p^2\right)k\cdot p\right)F'(k+p)\right)\right)\right)\right)\\
&+\left(3p\mu(p)-3p\mu(k+p)+2MF(p)k\cdot pF'(p)\right)\left(F(k+p)\left(pF'(p)+k\cdot pF''(p)\right)\right)\\
&-\frac{1}{(k+p)^3}\left(F(p)\left(\sqrt{(k+p)^2}(k^2+k\cdot p)(p^2+k\cdot p)F''(k+p)\right.\right.\\
&\left.\left.-(p^4+2k^2p^2+(k\cdot p)^2+(k^2+3p^2)k\cdot p)F'(k+p)\right)\right)))]\,. \qquad \text{(B1)}
\end{aligned}
$$

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